Pacific Islands Fisheries Science Center

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INTRODUCTION

The purpose of this report is to provide the information needed to calculate unbiased estimates of the parameters of a length-weight relationship for a given sample of length-weight data from a fish species using the method of maximum likelihood.

MATERIALS AND METHODS

Here is the standard allometric equation to predict fish weight (*W*) at length (*L*).

$$W \;=\; A\cdot L^{B} \tag{1.1}$$

In equation (1.1), the parameters *A* and *B* are to be estimated with the available length-weight data. The parameter *A* is a scaling coefficient for the weight at length of the fish species. The parameter B is a shape parameter for the body form of the fish species. In theory, one might expect that the exponent B would have a value of roughly $B = 3$ because the volume of a 3-dimensional object is roughly proportional to the cube of length for a regularly shaped solid. For example, the volume (*V*) of a square box with sides of length *L* is $V = L^3$. In practice, fish that have thin elongated bodies will tend to have values of B that are less than 3 while fish that have thicker bodies will tend to have values of B that are greater than 3.

It is assumed that the length-weight data for the fish species consist of a total of *n* length and weight measurements from individual fish. That is, the length-weight data set (*D*) consists of the weight-length measurements $D = \{(W_1, L_1), (W_2, L_2), \ldots, (W_n, L_n)\}$ where W_k is the weight of the k^{th} fish and L_k is the weight of the k^{th} fish.

Note that equation (1.1) is nonlinear and there is no direct solution for the parameters *A* and *B* that produced an observed data set *D*. However, if one transforms the allometric equation by applying the natural logarithm to both sides of equation (1.1), then a linear regression equation to predict the logarithm of weight as a function of the logarithm of length and the transformed parameters can be derived:

$$\log W \;=\; \log A \;+\; B\cdot\log L \;\equiv\; b0 \;+\; b1\cdot\log L + \varepsilon \tag{1.2}$$

The log-transformed equation (1.2) is a linear regression model with an intercept parameter $b0$ and slope parameter $b1$ along with a normally distributed error term ε that has an expected value of zero and a constant variance. In particular, note that the transformed parameters are $b0 = \log A$ and $b1 = B$. The linear regression model in equation (1.2) can be fit to the observed length-weight data using the method of maximum likelihood to obtain maximum likelihood estimates (MLEs) of the parameters $b0$ and $b1$, where each data point is fit with a residual error ε that represents the difference between the observed weight value and the predicted weight using the estimated regression parameters.

The definition of the residual error for the kth fish (ε_k), which is equal to the logarithm of the observed fish weight minus the predicted logarithm of weight of the k^{th} fish, is

(1.3) $$\varepsilon_k = \log W_k - \left(b0 + b1 \cdot \log L_k\right)$$

If the linear regression model is fit to the length-weight data using the method of maximum likelihood by solving the normal equations (see, for example, Larsen and Marx, 1981), analytical estimates of the parameters $b0$ and $b1$ can be derived. In particular, if the errors in the predicted log-transformed weight from the linear model are normally distributed with constant variance, i.e.,

(1.4) $$\log W_k \sim N\left(b0 + b1 \cdot \log L_k , \sigma^2\right)$$

then the maximum likelihood estimates of $b1$, $b0$, and σ^2 have exact solutions.

To express the exact solutions for the MLEs succinctly, denote the expected values of the log-transformed observed fish weights ($E[\log W]$) and lengths ($E[\log L]$) as

(1.5) $$E\left[\log W\right] = \frac{1}{n}\sum_{k=1}^{n}\log W_k \quad and \quad E\left[\log L\right] = \frac{1}{n}\sum_{k=1}^{n}\log L_k$$

Given these definitions, the maximum likelihood estimate of $b1$ is

(1.6) $$\widehat{b1} = \frac{\sum_{k=1}^{n}\left(\log L_k - E\left[\log L\right]\right)\cdot\left(\log W_k - E\left[\log W\right]\right)}{\sum_{j=1}^{n}\left(\log L_j - E\left[\log L\right]\right)^2}$$

and the maximum likelihood estimate of $b0$ is

(1.7) $$\widehat{b0} = E\left[\log W\right] - \widehat{b1}\cdot E\left[\log L\right]$$

and the bias-corrected maximum likelihood estimate of σ^2 is

(1.8) $$\widehat{\sigma^2} = \frac{1}{n-2}\sum_{k=1}^{n}\varepsilon_k^2$$

Maximum likelihood estimates of the variances of the parameters $b1$ and $b0$ can also be derived and are functions of σ^2. The variance of the slope parameter $b1$ is

$$(1.9) \qquad \widehat{VAR[b1]} = \frac{\sigma^2}{\sum_{k=1}^{n} \left(\log L_k - E[\log L]\right)^2}$$

and the variance of the intercept parameter *b*0 is

$$(1.10) \qquad \widehat{VAR[b0]} = \frac{\sigma^2 \cdot \sum_{k=1}^{n} \left(\log L_i\right)^2}{n \cdot \sum_{j=1}^{n} \left(\log L_i - E[\log L]\right)^2}$$

These variances can be used to construct confidence intervals for the parameters *b*0 and *b*1, using the standard deviations of *b*0 and *b*1 where

$$(1.11) \qquad \widehat{STDEV[b0]} = \sqrt{\widehat{VAR[b0]}} \ \ and \ \ \widehat{STDEV[b1]} = \sqrt{\widehat{VAR[b1]}}$$

Given the MLEs of the regression parameters *b*0 and *b*1, the MLE of the exponent parameter *B* for the original allometric equation is simply

$$(1.12) \qquad \widehat{B} = \widehat{b1}$$

The standard deviation of *B* is simply equal to the standard deviation of *b*1. That is

$$(1.13) \qquad \widehat{STDEV[B]} = \widehat{STDEV[b1]}$$

The MLE of the parameter *A* needs to be back-transformed from the logarithmic scale to obtain the parameter value in the original scale. The naive estimate of *A* is *A* = exp(*b*0). It can be shown that this estimate has a negative bias (Hayes et al., 1995). That is, the expected value of the naive estimate is less than the true value of *A*. This negative bias results from the fact that the regression was based on log-transformed (Miller, 1984). In particular, the basis for the linear regression model changes from the arithmetic mean in the original data units to the geometric mean in log-transformed units. The negative bias can be approximately corrected by multiplying the *A* parameter by $\exp(0.5\sigma^2)$, where σ is the estimated residual variance of the regression model fit (Hayes et al., 1995). Thus, the bias-corrected *A* parameter is

$$(1.14) \qquad \widehat{A} = \exp\left(\widehat{b0}\right)\exp\left(\frac{\widehat{\sigma^2}}{2}\right)$$

The bias-corrected standard deviation of *A* (STDEV[*A*]) can also be approximated in a similar manner as

(1.15) $$\widehat{STDEV[A]} = \exp\left(\sqrt{\widehat{VAR[b0]}}\right)\exp\left(\frac{\widehat{\sigma^2}}{2}\right)$$

The value of the coefficient of determination for the regression analysis (R^2) can be derived from the residuals of the regression fit as

(1.16) $$R^2 = 1 - \frac{\sum_{k=1}^{n} \varepsilon_k^2}{\sum_{j=1}^{n}\left(\log W_j - E[\log W]\right)^2}$$

The R^2 value provides a measure of the goodness-of-fit of the linear regression model to the length-weight data with higher R^2 values indicating better fits to the observed data.

SUMMARY

The allometric equation (equation 1.1) is a commonly used model to predict fish weights from fish lengths. Maximum likelihood estimates of the scale (*A*) and shape (*B*) parameters of the allometric equation can be calculated from a linear regression on log-transformed fish weight and length data. The analytical formulas for the maximum likelihood estimates of the scale parameter *A* and its variance have been provided (equations 1.6, 1.9, 1.10, 1.13, and 1.14), where a bias-correction factor has been included to adjust for transformation bias of the scale parameter. Similarly, the formulas for maximum likelihood estimates of the shape parameter *B* and its variance have also been provided (equations 1.5, 1.8, 1.11, and 1.12), and it is noted that no bias correction factor is needed for the shape parameter.

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